CONFORMAL SYMPLECTIC GEOMETRY OF COTANGENT BUNDLES

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Abstract. We prove a weak version of the Arnol'd conjecture for Lagrangian submanifolds of conformal symplectic manifolds: given a Lagrangian manifold so that the restriction of the Lee form has non-zero Novikov homology, we show that the time-1 flow of a C^2-small Hamiltonian cannot disjoin the Lagrangian from itself. We also give a short exposition of conformal symplectic geometry, aimed at readers who are familiar with (standard) symplectic or contact geometry.

1. Introduction

A *conformal symplectic structure* on a manifold M is a generalization of a symplectic structure. Locally, a conformal symplectic manifold is equivalent to a symplectic manifold, but the local symplectic structure is only well-defined up to scaling by a constant, and the monodromy of the local symplectic structure around curves may induce these rescalings. To our knowledge, the notion was first introduced by Vaisman in [20] and [21]. They also appeared in work of Guedira and Lichnerowicz in [12]. It was later studied by Banyaga [4], among many others.[1] More recently a first general existence result was given in [3] for complex surfaces with odd first Betti number, it was later proved in [10] that an almost symplectic manifold with non zero first Betti number admits a conformal symplectic structure, providing a large class of examples of such structures.

More formally, we can take a number of equivalent definitions. We can say that a conformal symplectic structure on M is an atlas of charts to $\mathbb{R}^{2n}$, so that the transition maps ψ_{ij} satisfy $\psi_{ij}^*\omega_{\mathrm{std}} = c_{ij}\omega_{\mathrm{std}}$, for constants $c_{ij} > 0$. Equivalently, let $E \to M$ be a flat, orientable, real line bundle. Then a conformal symplectic structure is a 2-form on M taking values in E, which is closed and non-degenerate.

Taking a connection on E leads to the most tractable definition. A *conformal symplectic structure* on M is a pair $(\eta, \omega) \in \Omega^1 M \times \Omega^2 M$, so that $d\eta = 0$, $d\omega = \eta\wedge\omega$, and $\omega^{\wedge n} \neq 0$. Because the choice of connection is non-canonical, (η, ω) defines the same conformal symplectic structure as $(\eta + df, e^f\omega)$, for any $f \in C^\infty M$. (The choice of a specific η representing $[\eta]$ is roughly analogous to a choice of contact form for a given contact structure, η is called a *Lee form* of the conformal structure as such form appeared in the work of Lee [13].)

For an example of a conformal symplectic manifold, let β be a closed 1-form on a manifold Q, let $\lambda = p \cdot dq$ be the tautological 1-form on T^*Q, let $\eta \in \Omega^1(T^*Q)$ be the pullback of β under the projection, and let $\omega = d\lambda - \eta \wedge \lambda$.

[1]In previous literature, this structure is often called a *locally* conformal symplectic structure, which is a more accurate term. But also more cumbersome.

Conformal symplectic manifolds enjoy many of the properties that make symplectic manifolds interesting. The definitions of Lagrangian, isotropic, etc. are exactly the same, and there is a natural notion of exact conformal symplectic structures, and exact Lagrangians inside them. They have a natural Hamiltonian dynamics (a smooth function defines a flow preserving the structure). They satisfy a Moser-type theorem, which implies that Darboux's theorem and the Weinstein tubular neighborhood theorems hold in this context (a small neighborhood of any Lagrangian is equivalent to the example above). When restricted to a coisotropic, the kernel of ω is a foliation, and in the case the leaf space is a manifold it inherits a conformal symplectic structure. The Poisson bracket on Hamiltonians intertwines the Lie bracket.

However, many of the more modern methods in symplectic geometry cannot be easily generalized, and often the theorems fail to hold true. For example, suppose that $\overline{\eta} \in \Omega^1 Q$ is a 1-form which never vanishes. Then in the conformal symplectic manifold (T^*Q, η, ω) defined above, the zero section $Z = \{p = 0\}$ is displaceable by Hamiltonian isotopy. In fact, if φ_t is the Hamiltonian isotopy generated by the Hamiltonian $H = 1$, then $\varphi_t(Z) \cap Z = \varnothing$ for any $t > 0$.

From the point of view of Floer theory, the problem with conformal symplectic structures is that we cannot even get started. The set of almost complex structures compatible with ω is still a non-empty contractible space, and $\overline{\partial}_J$ is still an elliptic operator, but because ω is not closed we have no bounds on energy, and therefore we do not expect Gromov compactness to hold, even for conformal symplectic manifolds which are both closed and exact. We discuss explicit examples suggesting failure of Gromov compactness in Section 2.5. Whether Gromov compactness can be generalized to this context by defining a more sophisticated compactification remains to be seen.

The main theorem of this paper proves a more modest result:

Theorem 1.1. *Let (M, η, ω) be a conformal symplectic manifold, and let $L \subseteq M$ be a Lagrangian submanifold. Suppose that the Novikov homology $H_*(L, [\eta]; \mathbb{F})$ is not zero, with coefficients in a field $\mathbb{F}$ (if L is not orientable one needs to assume $\mathbb{F} = \mathbb{Z}_2$). Then for any C^2 small Hamiltonian H, $\varphi^1_H(L) \cap L \neq \varnothing$.*

We also prove an analogous to Sikorav's Theorem [18] about persistence of generating families which allows to provide bounds for intersection of Lagrangian submanifolds in conformal contangent in terms of stable η-critical points of function. Unfortunately at the present time we have no example where this bounds leads to existence of intersection points (see Theorem 2.21).

The layout of the paper follows. Section 2 introduces the basic definitions and theorems in conformal symplectic geometry. We include a number of propositions which are not necessary for the proof of Theorem 1.1, with the hope of giving a large-scale introduction to the theory, particularly for symplectic and contact geometers.

Section 3 gives a brief overview of Morse-Novikov homology needed for the proof of Theorem 1.1. Finally, in Section 4 we complete the proof.

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2. Main definitions

2.1. Conformal symplectic manifolds.

Proposition 2.1. *Let M be a $2n$-manifold. The following are equivalent:*

- *An atlas of charts $M = \bigcup U_i$, $\varphi_i : U_i \to \mathbb{R}^{2n}$, so that the transition maps $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ preserve the standard symplectic form up to scaling by a positive local constant: $\psi_{ij}^*(\omega_{std}) = c_{ij}\omega_{std}$. Two atlases are considered equivalent if they admit a common refinement.*
- *A flat, real, orientable line bundle $E \to M$, and a 2-form $\sigma \in \Omega^2(M, E)$, so that σ is non-degenerate (as a map $TM \to T^*M \otimes E$) and closed (as a form with values in a flat line bundle). Two such structures (E_1, σ_1), (E_2, σ_2) are considered equivalent if there is an isomorphism $\varphi : E_1 \to E_2$ covering the identity map, so that $\varphi^*\sigma_1 = \sigma_2$.*
- *A pair $(\eta, \omega) \in \Omega^1 M \times \Omega^2 M$, so that $d\eta = 0$, $d\omega = \eta \wedge \omega$, and $\omega^{\wedge n} \neq 0$. (η, ω) is equivalent to $(\eta + df, e^f\omega)$ for any $f \in C^\infty M$.*

Any of these structures is called a conformal symplectic structure *on M.*

Proof. Given an atlas, the association of the positive number c_{ij} to every intersection $U_i \cap U_j$ can be thought of as the clutching function for a principal $GL^+(1, \mathbb{R})$-bundle on M, with the discrete topology. That is, the numbers c_{ij} define an orientable flat line bundle, E, and the form $\sigma = \varphi_i^*\omega_{\mathrm{std}}$ is well defined as a 2-form taking values in E. It is closed and non-degenerate, since these are local conditions.

Given a pair (E, σ), we can identify $E \cong M \times \mathbb{R}$ globally, as smooth vector bundles, being a real, orientable line bundle. A choice of flat connection on E then is simply a closed 1-form $\eta \in \Omega^1(M)$. In this way we can identify σ as an ordinary 2-form $\omega \in \Omega^2(M)$. Then to say that ω is closed as a 2-form with values in E is equivalent to the statement $d\omega - \eta \wedge \omega = 0$. The choice of connection η is not canonical; gauge symmetries of E act by $\eta \mapsto \eta + df$ for any $f \in C^\infty(M)$, and this gauge symmetry acts on ω by $\omega \mapsto e^f\omega$.

Given a pair (η, ω) as above, consider the covering space $\pi : \widetilde{M} \to M$ associated to $[\eta]$. Then $\pi^*\eta = d\theta$ for some $\theta \in C^\infty\widetilde{M}$, and for any covering transformation $g \in \pi_1 M/\ker[\eta] \subseteq \mathrm{Diff}\,\widetilde{M}$, we have $\theta \circ g = \theta + \langle[\eta], g\rangle$. Let $\widetilde{\omega} = e^{-\theta}\pi^*\omega \in \Omega^2\widetilde{M}$. Then $\widetilde{\omega}$ is a symplectic form, and $g^*\widetilde{\omega} = e^{-\langle[\eta],g\rangle}\widetilde{\omega}$. Therefore by taking a Darboux atlas on $\widetilde{M}$, we get a conformal symplectic atlas on M. □

Definition 2.2. Let η be a closed 1-form on a manifold M. The *Lichnerowicz-De Rham differential* on $\beta \in \Omega^*(M)$ is defined as $d_\eta\beta = d\beta - \eta \wedge \beta$.

From the fact that η is closed and of odd degree we get that $d_\eta^2 = 0$ and from the fact that η has degree 1 we see that d_η is degree 1 as well. Perhaps the most essential difference between d_η and d is that d_η does not satisfy a Stokes' theorem. We note a couple important formulas:

$$d_{\eta+df}\beta = e^f d_\eta(e^{-f}\beta)$$

$$\mathcal{L}_X\beta = X \lrcorner d_\eta\beta + d_\eta(X \lrcorner \beta) + \eta(X)\beta$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. By passing to the covering space of M defined by $\ker[\eta]$, it follows that the homology $H^*(\Omega^* M, d_\eta)$ is isomorphic to $H^*(M; [\eta], \mathbb{R})$, the homology of M with local coefficients defined by the homomorphism $[\eta] : \pi_1 M \to \mathbb{R}$.

Note. For concreteness, we will state definitions and prove propositions in the setup where a conformal symplectic structure is a pair (η, ω), and then when relevant show that the definitions are invariant under gauge equivalence $\eta \rightsquigarrow \eta + df$. By working directly with the complex $\Omega^*(M, E)$ for the flat line bundle E, the arguments are more elegant, though likely more opaque.

Definition 2.3. Let (M, η, ω) be a conformal symplectic structure. We say that a submanifold $L \subseteq M$ is *isotropic* if $\omega|_L = 0$, *coisotropic* if $TL^{\perp\omega} \subseteq TL$, and *Lagrangian* if it is isotropic and coisotropic.

We say that (η, ω) is *exact* if $\omega = d_\eta\lambda$ for some $\lambda \in \Omega^1 M$. In this case we say that λ is a *Liouville form* for (η, ω), and the vector field Z_λ defined by $Z_\lambda \lrcorner \omega = \lambda$ is called the *Liouville vector field.*[2] If $L \subseteq M$ is a Lagrangian in $(M, \eta, d_\eta\lambda)$, we say that L is *exact* if $\lambda|_L = d_\eta h$ for some $h \in C^\infty L$.

Notice that all of the above definitions are well defined up to gauge equivalence, because if we replace (η, ω) with $(\eta + df, e^f\omega)$, we can also replace λ and h with $e^f\lambda$ and $e^f h$. In particular, the Liouville vector field Z_λ is well defined up to gauge, it is *not* rescaled by e^f. Note also that there is nothing precluding the existence of a closed, exact conformal symplectic manifold.

Example 2.4. Let (Y, λ) be a manifold with a 1-form λ. We denote by S^1_T the quotient $\mathbb{R}/T\mathbb{Z}$ and assume it is parametrised by θ. On $S^1_T \times Y$ let $\eta = -d\theta$. Then $\omega = d_\eta\lambda = d\lambda + d\theta \wedge \lambda$, so ω is non-degenerate if and only if λ is a contact form. Furthermore, given another contact form defining the same cooriented contact structure, $e^f\lambda$, we have that the conformal Liouville manifold $(S^1_T \times Y, \eta, \lambda)$ is gauge equivalent to $(S^1_T \times Y, \eta + df, e^f\lambda)$, which is conformal symplectomorphic to $(S^1_T \times Y, \eta, e^f\lambda)$ under the coordinate change $(\theta, y) \mapsto (\theta - f(y), y)$.

The minimal cover which makes η exact is $\mathbb{R} \times Y \to S^1 \times Y$, and the conformal Liouville structure (η, λ) pulls back to $(-dt, \lambda)$, which is gauge equivalent to the exact symplectic structure $e^t\lambda$, known as the symplectization of $(Y, \ker\lambda)$. Hence, the conformal symplectic manifold will be called the *conformal symplectization* of (Y, ξ). We denote this manifold by $S^{\mathrm{conf}}_T(Y, \ker\lambda)$. (Notice that $\langle[\eta], S^1 \times \{\mathrm{point}\}\rangle = -T$, so the choice of T affect the conformal symplectomorphism type.) If the choice of T is irrelevant, we will use the notation $S^{\mathrm{conf}}(Y, \ker\lambda)$.

Now, let $\Lambda \subseteq Y$ be a Legendrian submanifold. Then the cylinder $\mathbb{R} \times \Lambda$ is an exact Lagrangian in the symplectization of $(Y, \ker\lambda)$, and furthermore it is invariant under the covering transformations of $\mathbb{R} \times Y \to S^1 \times Y$. Therefore it descends to an exact Lagrangian submanifold $S^1 \times \Lambda$ of $S^{\mathrm{conf}}(Y, \ker\lambda)$. More generally, if Σ is a Lagrangian cobordisms from Λ to itself which is cylindrical outside of $(0, T) \times Y$ then Σ descends to a Lagrangian submanifold of $S^{\mathrm{conf}}_T(Y, \ker\lambda)$.

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[2] This usage is somewhat different from the standard symplectic case, where a Liouville form is required to be a primitive of ω and also satisfy a convexity condition near the boundary/infinite portion of M. The same conditions on Z_λ make sense in this context, so they could easily be imposed.

This paper will focus primarily on the following example:

Example 2.5. Let β be a closed 1-form on a smooth manifold Q, let π be the projection $T^*Q \to Q$, and let λ_{std} be the tautological form on T^*Q. Define $\eta := \pi^*\beta$. Then $(T^*Q, \eta, \lambda_{\text{std}})$ is a conformal Liouville manifold (since $\eta \wedge \lambda_{\text{std}} \wedge d\lambda_{\text{std}}^{n-1} = 0$). We will denote this conformal Liouville manifold by $T^*_\beta Q$.

Let $\alpha : Q \to T^*Q$ be a 1-form. Then $\alpha^*\lambda_{\text{std}} = \alpha$ and thus $\alpha(Q)$ is Lagrangian in $T^*_\beta Q$ if and only if $d_\beta \alpha = 0$. Furthermore this Lagrangian is exact if and only if α is d_β-exact.

2.2. **Conformal symplectic transformations.** A diffeomorphism between conformal symplectic manifolds $\varphi : (M_1, \eta_1, \omega_1) \to (M_2, \eta_2, \omega_2)$ is called a *conformal symplectomorphism* if $\varphi^*\eta_2 = \eta_1 + df$ and $\varphi^*\omega_2 = e^f \omega_1$ for some $f \in C^\infty M_1$. We denote by $\text{Symp}(M, \eta, \omega)$ the group of conformal symplectomorphisms of (M, η, ω). The Lie algebra of this group is given by vector fields X satisfying $\mathcal{L}_X \omega = f\omega$ and $\mathcal{L}_X \eta = df$ for some f, and using Cartan's formula we see this is equivalent to

$$d_\eta(X \lrcorner\, \omega) + \eta(X)\omega = \mathcal{L}_X \omega = f\omega$$
$$d(\eta(X)) = \mathcal{L}_X \eta = df$$

Therefore X is a conformal symplectic vector field if and only if $d_\eta(X \lrcorner\, \omega) = c\omega$ for a constant $c \in \mathbb{R}$.

The subalgebra of $\mathfrak{S}\text{ymp}(M, \eta, \omega)$ ω-dual to η-closed 1-forms (i.e. vector fields X with $d_\eta(X \lrcorner\, \omega) = 0$) will be called *divergence free conformal symplectic vector field*. Notice the following fact: a conformal symplectic manifold is exact if and only if it admits a conformal symplectic vector field which is not divergence-free. In this case, after choosing a Liouville form λ, we see that every conformal symplectic vector field is of the form $X + cZ_\lambda$, where X is divergence-free.

Definition 2.6. The conformal symplectic vector fields which are dual to η-exact forms is called *Hamiltonian vector fields*. For any function $H \in C^\infty M$, the Hamiltonian vector field X_H associated to H is defined by $X_H \lrcorner\, \omega = -d_\eta H$. A conformal symplectomorphism of M which is the time-1 flow of a path of Hamiltonian vector fields will be called a *Hamiltonian diffeomorphism*.

Notice that the association $H \rightsquigarrow X_H$ depends on η, but the algebra of Hamiltonian vector fields (and therefore the group of Hamiltonian diffeomorphisms) does not. Using X_H^η to denote this dependence, we immediately have $X_H^\eta = X_{e^f H}^{\eta + df}$. Or to phrase the dependence in a gauge free way, if E is the flat line bundle determined by $[\eta]$, then Hamiltonians are taken to be $H \in \Omega^0(M, E)$, and since $\omega \in \Omega^2(M, E)$, the equation $X_H \lrcorner\, \omega = -d_E H$ defines X_H unambiguously.[3]

Definition 2.7. Given a conformal symplectic structure with a chosen connection η, we define the *Lee vector field* to be the Hamiltonian vector field generated by $H = 1$. We sometimes denote this vector field as $R_\eta = X_1$.

Example 2.8. Given a contact manifold $(Y, \ker \alpha)$, we defined the conformal symplectization in Example 2.4 by $(S^1 \times Y, \eta = -d\theta, \omega = d_\eta \alpha)$. In this case, the Lee vector field of η is equal to the Reeb vector field of α.

[3] The reader experienced with contact geometry will be familiar with this situation: the contact vector field X_H associated to a Hamiltonian $H \in C^\infty M$ depends on a choice of contact form, but the correct way to define the relationship without reference to a contact form is to take contact Hamiltonians $H \in \Omega^0(M, TM/\xi)$.

Note that this exhibit a first difference with the symplectic case: this flow has no fixed point for small time. This is an example of a more general phenomenon: given a conformal symplectic manifold (M, η, ω), if η can be chosen to have no zeros, then R_η is a Hamiltonian vector field with no zeros, and therefore we can construct many Hamiltonian diffeomorphisms with no fixed points. Another place where this condition is relevant is about Lagrangian displaceabality:

Example 2.9. Given a cotangent bundle $T^*_\beta Q$ with Liouville structure given by a closed one form β on Q as in Example 2.5, the Liouville vector field Z_λ is given by the standard Liouville vector field on T^*Q. The Lee vector field R_η corresponds to the symplectic vector field dual to η (in the standard symplectic sense), i.e. its flow acts fiberwise, and translates the vertical fiber $T^*_q Q$ in the direction β_q. In particular, if $[\beta] \in H^1 Q$ can be represented by a non-vanishing closed 1-form, then there exists an autonomous Hamiltonian flow on $T^*_\beta Q$ so that $\varphi^t_H(Q) \cap Q = \varnothing$ for *any* $t > 0$.

Given an exact conformal symplectic manifold $(M, \eta, d_\eta \lambda)$, there is some interplay between the vector fields Z_λ and R_η. Always, we have

$$\lambda(R_\eta) = \omega(Z_\lambda, R_\eta) = -\eta(Z_\lambda).$$

Furthermore, suppose that $X \in \ker d\lambda$. Since $\omega = d_\eta \lambda = d\lambda - \eta \wedge \lambda$, we get that $X \lrcorner \omega = \lambda(X)\eta - \eta(X)\lambda$, and therefore $X = \lambda(X) R_\eta - \eta(X) Z_\lambda$ since ω is non-degenerate. It follows that, if $d\lambda$ has non-zero kernel, it must be equal to the span of R_η and Z_λ. Plugging either R_η or Z_λ into the original equation, we see that $d\lambda$ has non-zero kernel if and only if $\eta(Z_\lambda) = 1$, if and only if $\lambda(R_\eta) = -1$.

In the literature, exact conformal symplectic manifolds satusfying $\eta(Z_\lambda) = 1$ everywhere are often referred to as conformal symplectic manifolds *of the first kind*. Notice that this condition is not gauge invariant, and it can be difficult to tell when a given conformal symplectic manifold is gauge-equivalent to one satisfying this property. However, this condition has strong implications, as shown in [6, Theorem A]: if M is compact and $\eta(Z_\lambda) = 1$ everywhere, and if $\{\eta = 0\}$ has at least one compact leaf, then M is symplectomorphic to the suspension of a strict contactomorphism of a contact manifold Y.

2.3. **Moser's theorem and tubular neighborhoods.** We have the following version of Moser's Theorem (first proved in [4]).

Theorem 2.10. *Let (M, η) be a closed smooth manifold equipped with a closed 1-form, and let ω_t $t \in [0,1]$, be a path of conformal symplectic structures in the same homology class, i.e. $\omega_t = \omega_0 + d_\eta \lambda_t$. Then there is an isotopy $\varphi_t : M \to M$ and a path of smooth functions $f_t \in C^\infty M$ so that $\varphi_t^* \eta = \eta + df_t$ and $\varphi_t^* \omega_t = e^{f_t} \omega_0$. That is, an exact homotopy of conformal symplectic structures is always induced by an isotopy (and gauge transformations)*

Proof. We find a vector field X_t suitable for $X_t = \dot{\varphi}_t \circ \varphi_t^{-1}$, and then using compactness of M integrate X_t. Differentiating the desired equations gives

$$\varphi_t^* \left(\mathcal{L}_{X_t} \eta\right) = d\dot{f}_t$$

$$\varphi_t^* \left(\dot{\omega}_t + \mathcal{L}_{X_t} \omega_t\right) = \dot{f}_t e^{f_t} \omega_0 = \dot{f}_t \varphi_t^* \omega_t.$$

Letting $\mu_t = \dot{f}_t \circ \varphi_t^{-1}$, we have

$$d(\eta(X_t)) = \mathcal{L}_{X_t} \eta = d\mu_t$$

$$\dot{\omega}_t + d_\eta(X_t \lrcorner \omega_t) + \eta(X_t)\omega_t = \mu_t\omega_t.$$

Defining X_t by $X_t \lrcorner \omega = \dot{\lambda}_t$, and taking $\mu_t = \eta(X_t)$, we see that this system has a solution. □

Just as in the symplectic case, the Moser theorem for conformal symplectic structures is used to prove a Weinstein neighborhood theorem for Lagrangians in conformal symplectic manifolds, showing that Example 2.5 is a universal model. More precisely we have the following result, which appears in the recent paper [15]. It also follows from a more general version for coisotropic manifolds, appearing in [22, Theorem 4.2].

Theorem 2.11. *Let (M, η, ω) be a conformal symplectic manifold and $L \subset M$ a compact Lagrangian. Let $\beta = \eta|_L$. Then there exists a neighborhood $\mathcal{N} \subseteq M$ of L so that $\mathcal{N}$ is conformally symplectomorphic to a neighborhood U inside $(T^*_\beta L, \pi^*\beta, d_{\pi^*\beta}\lambda_{std})$, sending $L \subseteq \mathcal{N}$ to the zero section $\{p = 0\} \subseteq U$.*

Proof. Letting $\mathcal{N}$ be a tubular neighborhood of L, we have a diffeomorphism $\psi : T^*L \to \mathcal{N}$ sending the zero section $Z \subseteq T^*L$ to $L \subseteq \mathcal{N}$, so that $\psi^*\omega|_{T_Z T^*L} = d_{\pi^*\beta}\lambda_{\mathrm{std}}|_{T_Z T^*L}$. $\psi^*\eta$ and $\pi^*\beta$ are homologous, so by taking a gauge transformation on $\mathcal{N}$ we can assume that they are equal.

Since $\psi^*\omega_1|_Z = 0$, therefore $\psi^*\omega = d_{\pi^*\beta}\lambda_1$ is exact (the Lichnerowicz-De Rham complex is homotopy invariant). We can furthermore assume that $\lambda_1|_{T_Z T^*L} = 0$ by addition of a $d_{\pi^*\beta}$-closed 1-form. Let $\lambda_t = t\lambda_1 + (1-t)\lambda_{\mathrm{std}}$, and $\omega_t = d_\eta\lambda_t$. By taking a smaller neighborhood if necessary, we may assume that ω_t is non-degenerate for all $t \in [0, 1]$.

We then run the Moser method, as above. The vector field X_t obtained is zero along Z, and therefore by shrinking $\mathcal{N}$ further if necessary, we get a conformal symplectomorphism from a neighborhood of $Z \subseteq T^*_\beta L$ to $\mathcal{N}$. □

2.4. **Flexibility and rigidity.** Given a closed manifold M^{2n}, we would like to know when M admits a conformal symplectic structure. If so, how many distinct structures are there up to isotopy. There are some basic obstructions/invariants. For ease of notation we focus on the exact case.

Definition 2.12. Let M^{2n} be a closed manifold. An *exact almost conformal symplectic structure* (or EACS structure) is a pair (a, w), where $a \in H^1(M; \mathbb{R})$, and $w \in \Omega^2 M$ is a non-degenerate 2-form.

Any exact conformal symplectic structure $(\eta, d_\eta\lambda)$ defines an almost conformal symplectic structure with $a = [\eta]$, $w = d_\eta\lambda$. Therefore, for a manifold to admit an exact conformal symplectic structure, it must also admit an EACS structure. Similarly, for two conformal symplectic structures to be isotopic, they necessarily must be homotopic through EACS structures (where w is homotoped but a is fixed). The benefit of passing to EACS structures is that they are classified purely by algebraic topology.

We show that that the classification of exact conformal symplectic structures is strictly more subtle than their almost conformal counterpart.

Proposition 2.13. *There exists a manifold M of any dimension and and $[\eta] \neq 0 \in H^1M$, which admits two exact conformal symplectic structures $(\eta, d_\eta\lambda_1)$ and $(\eta, d_\eta\lambda_2)$ which are homotopic through non-degenerate 2-forms, but they are not conformally symplectomorphic.*

Proof. We take $M = S^1 \times S^{2n-1}$. Let $\xi_{\mathrm{ot}} = \ker \alpha_{\mathrm{ot}}$ be an overtwisted contact structure (see [7]) on S^{2n-1}, which is homotopic through almost contact structures to the standard contact structure ξ_{std}.

As the symplectization of $(S^{2n-1}, \xi_{\mathrm{ot}})$ is different form the one of $(S^{2n-1}, \xi_{\mathrm{std}})$ (actually the second cannot embed in the first as $(S^{2n-1}, \xi_{\mathrm{std}})$ is fillable) we know that those two structure are not equivalent.

Moser's theorem therefore implies that they are not homotopic through exact conformal symplectic structures. But they are homotopic through non-degenerate 2-forms, since the original contact structures are homotopic through almost contact structures. □

Remark 2.14. As shown in [14], when $n > 2$ the symplectization $(\mathbb{R}\times S^{2n-1}, e^r\alpha_{\mathrm{ot}})$ contains an exact Lagrangian, $L \subseteq \mathbb{R} \times S^{2n-1}$. Let $R > 0$ be a constant so that $L \subseteq (0, R) \times S^{2n-1}$. Then by identifying $S^1 = \mathbb{R}/R\mathbb{Z}$ L embeds as an exact Lagrangian into the conformal symplectization $(M, \eta = -d\theta, \lambda_1 = \alpha_{\mathrm{ot}})$.

However, there can be no nulhomologous exact Lagrangian in the conformal symplectization of the standard contact structure, $(S^1 \times S^{2n-1}, \eta, \lambda_2 = \alpha_{\mathrm{std}})$. For if there was, it would lift to a compact exact Lagrangian in the universal cover, which is conformally symplectomorphic to $\mathbb{C}^n \setminus \{0\}$, which then contradicts Gromov's theorem. This shows that those two conformal symplectic structure can be distinguished by their Lagrangian submanifolds.

For $2n = 2$ we know that conformal symplectic structures are equivalent to almost conformal symplectic structures, since all 2-forms are exact when $[\eta] \neq 0$.

On the question of existence, a nearly complete answer was established in [10]:

Theorem 2.15. *Let M be a closed manifold with an EACS structure (a, w), where $a \neq 0$ is in the image $H^1(M;\mathbb{Z}) \to H^1(M,\mathbb{R})$. Then for all sufficiently large $C \in [1, \infty)$, there is an exact conformal symplectic structure $(\eta, d_\eta\lambda)$, with $[\eta] = Ca$ and $d_\eta\lambda$ being homotopic to w through non-degenerate 2-forms.*

In particular, given any closed manifold M satisfying $H^1(M;\mathbb{R}) \neq 0$ and having a non-degenerate 2-form, M admits an exact conformal symplectic structure.

This follows indeed from the existence result of symplectic structure on almost symplectic cobordisms by cutting along an hypersurface Poincaré dual to $[\eta]$ which is non-separating by hypothesis. Making then the cobordism symplectic between the same contact structure on the ends leads to a symplectic cobordisms whose ends can be identified to gives a conformal symplectic manifold.

It is an open question whether we can in general construct conformal symplectic structures where $[\eta]$ is a prescribed 1-form, either in the case where $[\eta]$ is small, or when $[\eta] : \pi_1 M \to \mathbb{R}$ has non-discrete image. Note that in [3], there is one example that shows that after fixing a complex structure J, the set of $[\eta]$ for which there exists a conformal symplectic structure compatible with J consists of a single point.

Remark 2.16. From this theorem, it is easy to construct non-exact conformal symplectic structures: given an exact symplectic structure $(\eta, d_\eta\lambda)$ and a class $b \in H^2(M; [\eta], \mathbb{R})$, choose a d_η-closed 2-form $B \in \Omega^2 M$ representing b. Then for sufficiently large C, $\omega = Cd_\eta\lambda + B$ is a conformal symplectic structure.

Remark 2.17. The same strategy also works for constructing conformal symplectic manifolds with convex contact boundary: any contact manifold which admits an almost complex filling admits a conformal symplectic filling. Indeed from such a formal filling the main theorem in [10] allows us to construct a symplectic cobordisms from $(S^{2n-1}, \xi_{\mathrm{ot}})$ to $(S^{2n-1}, \xi_{\mathrm{ot}}) \sqcup (Y, \xi)$, and gluing the two $(S^{2n-1}, \xi_{\mathrm{ot}})$ leads to a conformal symplectic filling of (Y, ξ).

As stated there, the theorem only applies to cobordisms with tight convex boundary when $n > 2$. However, after an examination of the proof there it is clear that the result applies when $n = 2$ as well, as long as a single component of the convex boundary is overtwisted.

2.5. **Pseudo-holomorphic curves and a "failure" of Gromov compactness.** Inspired from successes in symplectic and contact geometry, we might try to develop a theory of pseudo-holomorphic curves for conformal symplectic manifolds. Similar to the symplectic case, in a conformal symplectic manifold the space of compatible almost complex structures J is contractible, and the equation $\overline{\partial}_J u = 0$ is elliptic. But when trying to prove a compactness theorem, we run into an immediate problem: the Lichnerowicz-De Rham complex does not satisfy Stokes' theorem, and without this we have no way to give a C^0 bound on holomorphic energy. We give here two examples of holomorphic curves arising naturally which suggest bad compactness properties.

Consider the conformal sympectisation of an overtwisted sphere $Y = S^{2n-1}_{\mathrm{ot}}$ with a generic contact form. When equipped with a cylindrical almost complex structure J, there is a holomorphic plane which is asymptotic to one of the Reeb orbits, as shown in [2]. In the conformal symplectization $S^{\mathrm{conf}}(Y)$ this gives a holomorphic plane, which looks something like a leaf of a Reeb type foliation. Though such curves are handled in the symplectization setting by SFT compactness [8] and [1], this sort of compactness strongly relies on having contact asymptotics and a cylindrical complex structure, which is not a natural notion in general conformal symplectic manifolds.

A second phenomena is given by the conformal symplectic filling of S^3_{ot} described in the previous section. The bishop family near the elliptic point of an overtwisted disk leads to a family of holomorphic curves which cannot be compactified in a standard way. We suspect that such a family breaks into curves involving half-plane such as those described before.

2.6. **Contactisation, reduction, and generating functions.**

Definition 2.18. If (M, η, λ) is a conformal Liouville manifold, we define the *contactisation* of (M, η, λ) to be the manifold $M \times \mathbb{R}$ equipped with the contact structure given by the kernel of $\alpha = dz - z\eta - \lambda = d_\eta z - \lambda$ (note that α is a contact form if and only if $d_\eta \lambda$ is non-degenerate).

Notice that an exact Lagrangian submanifold L lifts to a Legendrian submanifold $\widetilde{L} = \{(q, f(q)); q \in L\}$ where f is such that $d_\eta f = \lambda|_L$.

In the cotangent case, the space $\mathcal{J}^1_\beta(Q) := T^*_\beta Q \times \mathbb{R}$ with the contact form $d_\beta z - \lambda$ is called the *β-jet bundle*, natural Legendrian submanifolds arise as β-jets of functions f, i.e. $j^1_\beta f(q) = (q, d_\beta f_q, f(q))$. Note that $\mathcal{J}^1_\beta(Q)$ is actually contactomorphic to $\mathcal{J}^1(Q)$ by the map $\varphi(q, p, z) = (q, p - z\beta_q, z)$. This contactomorphism is not compatible with the projection to T^*Q and therefore does not identify the symplectic properties of T^*Q with the conformal symplectic properties of $T^*_\beta Q$. Note however that it maps β-jets of functions to (regular) jets of functions.

The Reeb vector field of α is given by $R_\alpha = (1 + \lambda(R_\eta))\partial_z - R_\eta$.

Example 2.19. Let (M, α) be a contact manifold. Then the contact form on $M \times S^1 \times \mathbb{R}$ is $dz - zd\lambda - \alpha$ the Reeb vector field is just the original Reeb vector field seen as vector fields invariant by $S^1 \times \mathbb{R}$. The contactization of a conformal symplectization is a contact manifold associated to $(M, \ker \alpha)$, this manifold is the product of the original contact manifold and T^*S^1 with the Liouville structure given by $pd\theta - dp$.

Let (M, η, ω) be a conformal symplectic manifold. And let $Q \subset M$ be a coisotropic submanifold (i.e. such that $TQ^{\perp\omega} \subset TQ$). Since $d\omega = \eta \wedge \omega$ it follows from the Fröbenius integrability theorem that the distribution $\ker(\omega|_Q)$ is integrable. Note that for a vector field X in $\ker Q^*\omega$ we have that

$$\begin{aligned} L_X\omega &= \eta(X)\omega \\ L_X\eta &= d(\eta(X)). \end{aligned} \tag{2.1}$$

Let Q^ω be the leaf space of the associated foliation and assume that Q^ω is a manifold. We denote by p the projection from Q to Q^ω. It follows from equation (2.1) that on charts of Q^ω there is a well define conformal symplectic structure and thus that Q^ω is a conformal symplectic manifold. This conformal symplectic manifold is called the *conformal symplectic reduction* of Q.

Example 2.20. Let Q, Q' be two manifolds and β, β' be closed one form on Q, Q'. And let $T^*_{\beta\oplus\beta'}(Q \times Q') \simeq T^*_\beta Q \times T_{\beta'}Q'$ be the conformal symplectic manifold associated to $\pi^*\beta \oplus (\pi')^*\beta'$. Then $T^*_\beta Q \times Q'_0$ is cositropic and its reduction gives back $T^*_\beta Q$.

Consider a Lagrangian submanifold L of M and that Q intersects L cleanly. And denote by φ the projection $Q \to Q^\omega$. Then the manifold $L_Q = \varphi(L \cap Q)$ is an immersed Lagrangian submanifold of Q^ω. (This is linear algebra and thus equivalent to the symplectic case and follows from[5, Section 5.1] for instance).

Now given a map $F : Q \times Q' \to \mathbb{R}$, we can take the graph to get a Lagrangian section $d_{\beta\oplus\beta'}F$ of $T^*_{\beta\oplus\beta'}(Q\times Q')$, and apply symplectic reduction as above to obtain an immersed exact Lagrangian submanifold of $T_\beta Q$. When $Q' = \mathbb{R}^N$ (and $\beta' = 0$) and F is quadratic at infinity we denote the corresponding Lagrangian L_F. F is called a *β-generating family* for L_F.

Note that L_F lifts to a Legendrian submanifold Λ_F of $\mathcal{J}^1_\eta(Q)$ and through the contactomorphism ϕ defined above we get a Legendrian submanifold of $\mathcal{J}^1(Q)$, and F is a generating family (in the standard sense) for $\phi(\Lambda_F)$. Chekanov persistence's Theorem [9] therefore implies that if L in $T^*_\beta Q$ admits a generating family and L_t is an isotopy of L through exact Lagrangians then L_1 admits a β-generating family as well.

We remark that if L is given by a β-generating family $F : Q \times \mathbb{R}^N$ then zeros of $d_\eta F$ (called η-critical points of F) corresponds to intersection points between L and the zero section. All together, this implies

Theorem 2.21. *Let β be a closed 1-form on a closed manifold Q, and define $\mathrm{Stab}_\beta(Q) = \min_F \#\{q \in Q | d_\beta F(q) = 0\}$, where F is taken among all functions $F : Q \times R^N \to \mathbb{R}$ which are quadratic at infinity, for all $N \in \mathbb{N}$. That is, $\mathrm{Stab}_\beta(Q)$ is the stable β-critical number of Q. Let $L \subseteq T^*_\beta Q$ be a Lagrangian which is Hamiltonian isotopic to the zero section Z. Then the number of intersections between L and Z is at least $\mathrm{Stab}_\beta(Q)$.*

To get effective lower bound for $\#\{L \cap Z\}$, a natural thing to do would be to define a homology theory whose chains are linear combinations of $\{q|d_\beta F(q) = 0\}$, and then get lower bounds on the rank of this homology which do not depend on F. However, we do not know of such a homology theory, even in the easiest case where $F : Q \to \mathbb{R}$ and $d_\beta F$ is non-degenerate.

The homology theory in question one would like to define is something like the Morse-Novikov homology of $\log(|F|)$, but of course F can be zero. The rest of the paper explains a way to get around this issue in a particular case.

3. Overview of Morse-Novikov homology

In this section we present the basics of the construction of the Morse-Novikov complex associated to a closed 1-forms on Q. We also recall its essential properties we need to prove Theorem 1.1. We refer the reader to more comprehensive references like [11] and [16] for more details.

We assume that Q is connected. A closed 1-forms η is *Morse* if near each of its zeros it is the derivative of a non-degenerate quadratic form h. It follows from Morse's Lemma that any 1-forms whose graph intersect the 0-section transversely is Morse, therefore closed 1-forms are generically Morse. We refer to a zero of a closed 1-form η as a *critical point* of η. Given a critical point q of η the number of negative eigenvalues of the quadratic form h such that near q $\eta = dh$ is independent of the coordinate system, this number is called the *index* of q written $I_\eta(q)$.

Given a metric g on Q we define the vector field $\nabla\eta$ to be the dual of η with respect to g. We denote by x_t^η the flow of $\nabla\eta$. We will always assume that there is a compact subset K such that

- All critical points of η are in K (in particular there are finitely many critical points),
- $\nabla\eta$ is complete, so x_t^η is defined for all $t \in \mathbb{R}$,
- For every component V of $Q \setminus K$ either for all $q \in V$ and $t \geq 0$ $x_t^\eta(q) \notin K$ or for all $q \in V$ and $t \leq 0$ $x_t^\eta(q) \notin K$.

Any critical points q have stable and unstable manifolds $W^s(q)$ and $W^u(q)$ defined by $W^s(q) = \{q' | \lim_{t\to\infty} x_t^\eta(q') = q\}$ and $W^u(q) = \{q' | \lim_{t\to-\infty} x_t^\eta(q') = q\}$.

Note that $W^u(q)$ is a disk of dimension $I_\eta(q)$ and $W^s(q)$ is a disk of dimension $n - I_\eta(q)$. We say that the pair (η, g) is *Morse-Smale* if for any critical points q, q' of η the disks $W^u(q)$ and $W^s(q')$ intersects transversely. In this situation $W^s(q) \cap W^u(q')$ are open manifolds of dimension $I_\eta(q) - I_\eta(q')$ with an action of $\mathbb{R}$ on them which is free (unless $q = q'$).

We denote by $\Lambda_\eta(Q, R)$ the completion of the group ring $R[\pi_1(Q)]$ given by series of the form $\sum_i a_i g_i$ where $\lim_{i\to\infty} \int_{g_i} \eta = -\infty$.

Now for any critical points q of η we choose

(1) A path g_q from q to the base points.
(2) An orientation of the tangent space of q (we do not assume Q is orientable).

Given two critical points q, q' of η such that $I_\eta(q) - I_\eta(q') = 1$ then any component γ of $W^s(q) \cup W^u(q')$ is copy of $\mathbb{R}$ which compactifies to a path from q to q' which, together with the capping paths g_q and $g_{q'}$, gives an element g_γ of $\pi_1(Q)$.

This gives the series $u_{q,q'} = \sum_\gamma \pm g_\gamma$, which is in $\Lambda_\eta(Q, R)$ since we follow negative trajectories of $\nabla\eta$. The sign $\pm$ is determined by whether or not the chosen orientation of $T_q Q$ transports to the one of $T_{q'} Q$ along the path γ.

The Morse-Novikov complex of (η, g) is given by $C_k(\eta, R) = \oplus_{I_\eta(q)=k}\Lambda(Q,R)\langle q\rangle$ with differential $d(q) = \sum_{I_\eta(q')=k-1} u_{q,q'}q'$. We have that $d^2 = 0$ and the homology of (C_*, d) is called the Morse-Novikov homology of η denoted $H_*^{\mathrm{Nov}}(\eta)$.

When Q is compact it follows from a Theorem of Sikorav in [19] (see also [17]) that this homology is the homology of Q with local coefficients in $\Lambda_\eta(Q, R)$. Notice that if Q is compact with boundary and if $\nabla\beta$ points inward the boundary then the same results holds.

When R is a field, $\Lambda(Q, R)$ is a principal ring and we call the $k-th$ *Novikov-Betti number* b^k_η of η the rank of the free part of $H^{nov}_k(\eta)$. For compact oriented closed manifold they satisfy $b^k_\eta = b^{n-k}_\eta$ (see [16, Corollary 2.9] and [11, Section 1.5.3]), for non-oriented on the same is true if $R = \mathbb{Z}_2$.

4. Rigidity of Lagrangian intersection

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. As discussed in Section 2.3 it suffices to consider the case where $M = T^*_\beta Q$, and wish to show that $L \cap Z \neq \varnothing$, where $Z \subseteq T^*_\beta Q$ is the zero section and $L = \varphi^1(Z)$. We can assume the intersection is transverse (since this can be achieved by a C^2-small perturbation), and since H is C^2-small it follows that L is graphical and so it is equal to the image of $d_\beta f$ for some $f \in C^\infty Q$. $L \cap Z$ corresponds to the β-critical points of f, therefore it suffices to show that every $f \in C^\infty Q$ has a β-critical point.

Up to a small deformation we can assume that 0 is a β-regular value of f. We decompose Q as $X^+ \cup X^0 \cup X^- = \{f > 0\} \cup \{f = 0\} \cup \{f > 0\}$. The zeros of $d_\beta f = df - f\beta$ on X^+ are the zeros of $\beta - d(\log f)$, and on X^- they are the zeros of $\beta - d(\log(-f))$. Since we assume that 0 was a β-regular value there are no zeros on X^0.

Let $\mathcal{N}(X_0)$ be a neighborhood of X_0, and choose coordinates $(q, t) \in X_0 \times (-\varepsilon, \varepsilon)$ on $\mathcal{N}(X_0)$ so that $f(q, t) = t$. Choose a Morse-Smale metric so that these coordinates are orthogonal, and therefore outside of X_0 $\nabla \log |f| = \frac{\partial_t}{t}$. Let C be a constant larger than both $\frac{1}{\epsilon}$ and the uniform norm of β, and let $X_C = \{|t| < \frac{1}{C+1}\} \subseteq \mathcal{N}(X_0)$. Choose $g \in C^\infty M$ so that $g = \log |f|$ everywhere in the complement X_C, and $\beta - dg$ is Morse.

By construction the vector field V metric dual to $\beta - dg$ is pointing inward on the boundary of X_C. This implies that the Morse-Novikov complex $C_*^{X_C}$ of $(\beta - dg)|_{X_C}$ is well defined and is a subcomplex of the Morse-Novikov complex C^M_* of $(\beta - dg)$. We denote the quotient complex by $C_*^{X^+\cup X^-}$. By construction, this quotient complex is generated by zeros of $d_\beta f$.

The homology of $C_*^{X_C}$ is isomorphic to the Morse-Novikov homology of the pair $(X_0, (\beta - dg)|_{X_0})$, by homotopy invariance of Novikov homology. Since X_0 is compact, its (free) Betti numbers b^0_k satisfy Poincaré duality:

$$b^0_k = b^0_{n-1-k}, \tag{4.1}$$

and similarly for the Betti numbers b^M_k of C_M:

$$b^M_k = b^M_{n-k}. \tag{4.2}$$

In particular since b^M_k are not all 0 by hypothesis we have that there exist k such that $b^M_k \neq b^0_k$.

The result follows from the exact triangle associated to the sequence

$$0 \to C_*^{X_0} \to C_*^M \to C_*^{X^+\cup X^-} \to 0$$

$$\begin{tikzcd} H_*^{\mathrm{Nov}}(X_0, \beta - dg|_{X_0}) \arrow[rr, "i"] & & H_*^{\mathrm{Nov}}(M, \beta - dg) \arrow[dl] \\ & H_*(C_*^{X^+\cup X^-}) \arrow[ul, "{[-1]}"] & \end{tikzcd} \tag{4.3}$$

Indeed since i cannot be an isomorphism in all degrees we have that $H_*(C_{X^+\cup X^-}) \neq 0$ and thus $d_\beta f$ cannot be non-zero everywhere. □

Remark 4.1. In general it seems that the proof above cannot provide a lower bound $\#\{d_\beta f = 0\} \geq \dim H^{\mathrm{Nov}}(M, \beta; \mathbb{R})$, since the Novikov homology of X_0 depends on the choice of f. Under more hypotheses on the Novikov homology of β it is possible to obtain slightly more. For instance, assuming that there exists k such that $b_k^M = 0$ but $b_{k-i}^M \neq 0$ for some $i \in \mathbb{N}$ (and thus $b_{n-k}^M = 0$ and $b_{n+i-k}^M \neq 0$) the graded version of 4.3 provides more generators of $C_{X^+\cup X^-}$.

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